Designing Phase Shift Oscillators for Tremolo Circuits

What is a phase shift oscillator?

"Phase shift oscillator" is the term given to a particular oscillator circuit topology that uses an RC network in the feedback loop of a tube, transistor, or opamp to generate the required phase shift at a particular frequency to sustain oscillations. They are moderately stable in frequency and amplitude, and very easy to design and construct.

Where are they used?

Phase shift oscillators are most commonly used in tremolo circuits in guitar amplifiers. They are used as the low-frequency oscillator (LFO) that generates the sinusoidal waveform which amplitude modulates the guitar signal to produce the characteristic tremolo amplitude variations.

How do they work?

In order to create and sustain an oscillation at a particular frequency, a circuit must have a gain higher than unity, and a total phase shift around the loop of 360 degrees (which is equivalent to 0 degrees, or positive feedback). When used with a single-stage inverting amplification element, such as a tube, transistor, or inverting opamp configuration, the amplifier itself provides 180 degrees of phase shift (a gain of -A, where A is the gain of the amplification stage). The remaining 180 degrees of phase shift necessary to provide a total of 360 degrees is provided by an external network of resistors and capacitors.

Following is a schematic diagram of a typical phase shift oscillator:
Phase Shift Oscillator

The triode is configured as an inverting amplifier to provide the necessary gain, and the feedback network is connected from the plate to the grid.

The phase shift elements are C1/R1, C2/R2, and C3/R3. Three of these phase lead networks contribute a total of 180 degrees of phase shift at the oscillation frequency. Note that a phase shift oscillator could also be built using four or more phase shift elements, with each element contributing less overall phase shift at the oscillation frequency. Normally, there is no need to do this, as it takes extra components.

A minimum of three phase shift networks is required, however, because the maximum theoretical phase shift available from any one RC network is 90 degrees, and the actual phase shift approaches this value asymptotically.

A phase shift oscillator can also be made using three phase lag networks, which are obtained by swapping the positions of the R and C value components in the above schematic. The lag network would require one additional coupling cap to block the DC on the plate voltage from the grid, and one additional resistor to provide the grid bias ground reference for V1A, so it is not normally used.

Following is an example of both a phase lead and a phase lag network, designed for a 45 degree phase shift at the -3dB point of f = 1/(2*Pi*R*C) = 1/(2*Pi*1Meg*.01uF) = 15.9Hz:
Following is a plot of the phase shift and attenuation characteristics of the phase lead and phase lag networks:
As can be seen from the plot, the phase lead network starts at near +90 degrees at 0.1Hz, and shifts through +45 degrees at the -3dB point of 15.9Hz, continuing on toward 0 degrees above 1kHz. The phase lag network, on the other hand, starts at 0 degrees, shifts through -45 degrees at the -3dB point, and continues on towards -90 degrees above 1kHz. Either one will provide an effective 0 degrees phase shift when three of them are combined with the 180 degree phase shift of the amplifier as shown in the phase shift oscillator schematic.

It can be shown that the attenuation of the phase shift elements in the feedback loop is $1/29$, so the oscillator will oscillate if the amplifier gain is greater than 29 (which will bring the overall loop gain above unity gain, and satisfy the gain criterion for oscillation). The oscillations will occur at a frequency given by the following equation:

$$f_o = 1/(2\pi \sqrt{6RC})$$

In order to obtain the lowest distortion for the best sine wave, the amplifier should be operated with a gain of exactly 29, which is just the bare minimum necessary to sustain oscillation. This will produce the purest sine wave, however, it is impractical if tubes of varying gains may be
substituted (this usually requires an adjustment control to trim the gain), or if the frequency of oscillation must be adjusted in such a manner as to change the gain of the network. For these reasons, the gain is usually made higher, and post-filtering of the waveform is done to remove unwanted harmonic distortion.

If four phase lead networks are used, the phase shift per section at the oscillation frequency is lower, therefore, the attenuation of the network is also lower, around 1/18. This allows use of lower gain tubes if necessary, since the gain of the amplifier only has to be at least 18.

The design procedure

Following are the steps necessary to design a vacuum tube phase shift oscillator. For this example, we will assume that it is desired to make a three-stage vacuum tube phase shift oscillator with a nominal oscillation frequency of 7Hz.

- **Amplifier design:**

  - An amplifier with a gain of greater than 29 must first be designed. This requirement will affect the type of tube chosen. For example, a 12AU7 will not work in this circuit, because it has an amplification factor of only 17, which is lower than the minimum gain of 29. Note that the actual gain obtained from a tube is always considerably less than the amplification factor, which is the theoretical maximum obtainable from the tube, and then only with an infinite load impedance (a current source load is a good approximation to this). Typically, a 12AX7, with an amplification factor of 100, will achieve a gain of around 60 or higher with a resistive plate load.

  - Assuming a 12AX7 is chosen, following are the calculations for the gain and output impedance:

    Some typical 12AX7 numbers:

    - plate resistance: \( r_a = 62.5\, \text{K} \)
    - amplification factor: \( \mu = 100 \)

    - A plate load resistance of 100K and a cathode resistance of 820 ohms are chosen to provide good gain and linearity. In order to maximize the gain and minimize the output impedance, the cathode must be fully-bypassed at the frequency of oscillation. The midband gain for the 12AX7 with a fully-bypassed cathode is:

    \[
    A_v = \frac{\mu R_l}{R_l + r_a} \\
    = \frac{100 \times 100}{100 + 62.5} \\
    = 61.5
    \]

    Where:
    - \( R_l \) = the load resistor (100K in this example)
    - \( r_a \) = the internal plate resistance (62.5K for a typical 12AX7)
    - \( \mu \) = the mu of the tube (100 for a typical 12AX7)

    The calculated gain of 61.5 is higher than the required minimum gain of 29, so this amplifier will work in the phase shift oscillator circuit.
Next, the output impedance of the stage is calculated. Since the internal plate resistance is effectively in parallel with the plate load resistor, the output impedance (for the signal taken off the plate) will be:

\[ R = r_a \parallel R_l \]
\[ = 62.5K \parallel 100K \]
\[ = 38.5K \]

Note: the symbol "||" means "in parallel with". Resistors in parallel add in reciprocal, i.e. \(1/R_t = 1/R_1 + 1/R_2\).

Next, the resistance seen looking into the cathode must be calculated, as it is needed to determine the value of bypass capacitor needed.

The resistance seen looking into the cathode (with \(R_k\) unbypassed) is:

\[ R_k' = (R_l + r_a)/(\mu + 1) \]
\[ = (100K + 62.5K)/(101) \]
\[ = 1.6K \]

Therefore, the total cathode resistance is the parallel combination of the cathode resistance, \(R_k'\), and the cathode resistor, \(R_k\), as below:

\[ R = R_k' \parallel R_k \]
\[ = 1.6K \parallel 820 \]
\[ = 542 \text{ ohms} \]

Next, the value of the cathode bypass capacitor must be calculated. Since the gain of the amplifier must be maximized at the frequency of oscillation, the minimum capacitor value is chosen using the following equation for the -3dB point of the amplifier.

\[ f = 1/(2\pi R C) \]

solving for \(C\):

\[ C = 1/(2\pi R f) \]

The minimum value of bypass capacitor is therefore:

\[ C = 1/(2\pi * 542 * 7\text{Hz}) = 42\text{uF} \]

The amplifier gain will be down -3dB at this point, corresponding to a gain decrease of 0.707 times 61.5, or 43.5, which is still well above the required minimum of 29. However, in order to maximize gain and keep the phase shift associated with the cathode bypass capacitor to a minimum, the capacitor value should be increased to around five to ten times the minimum calculated value, such as 470\text{uF}, giving a -3dB point of 0.62Hz.

The bypass capacitor to achieve high gain at very low frequencies can get quite large if with small values of the cathode resistor. With
the 12AX7, there is enough excess gain that the circuit will still work, even at very low frequencies, but it may be a problem with other lower gain tubes.

- **Phase shift network component selection:**
  
  - Since the impedance is proportional to the shunt element in the phase shift network, in this case, the resistor, a suitable impedance value must first be chosen. The input impedance of the network must be large in comparison to the output impedance of the amplifier, so as to not load the output appreciably, which would reduce the gain, possibly to a point where it can no longer sustain oscillations. A good minimum value is around ten times the actual output impedance of the amplification stage. Since the input impedance is proportional to the shunt element, and is approximately twice the value of the shunt element at the oscillation frequency, the resistance can be chosen to be around half the required impedance. This resistance will then determine the value of capacitor necessary to achieve the desired frequency of oscillation. Since we have an output impedance of 38.5K in our example, a good minimum value for the input impedance is ten times this value, to prevent loading of the output stage. Since the resistance value to achieve this impedance is around half the total impedance, a value of five times the output impedance, or 5*38.5K = 193K, will work.

  - Next, the capacitance value is calculated using the formula for the frequency:
    \[
    f_0 = \frac{1}{(2\pi \sqrt{6})R \times C}
    \]

    solving for C:
    \[
    C = \frac{1}{(2\pi \sqrt{6})f_0 \times R} = 1/(2\pi \times 2.45 \times 7 \times 193K) = 0.048 \mu F \text{ (use 0.047uF as the nearest smaller standard value)}
    \]

    Since capacitor values are more commonly available in 10% values, and resistors are more commonly available in 5% or even 1% values, the resistor value should be recalculated based on the standard capacitor value chosen.

    solving for R:
    \[
    R = \frac{1}{(2\pi \sqrt{6})f_0 \times C} = 1/(2\pi \times 2.45 \times 7 \times 0.047uF) = 198K \text{ (use 200K as the nearest larger standard value)}
    \]

    When choosing the capacitor, it is best to choose the next smaller size because this will make the input impedance larger, because it will require a larger resistance to achieve the desired frequency. Likewise, when choosing the resistance, it is best to choose the next larger size, as this will also increase the input impedance of the phase shift network.

  - The design is then built and tested, and resistor or capacitor values are trimmed as necessary to provide the exact frequency of oscillation desired. Following is the output of the completed oscillator using the values calculated above:
A small amount of harmonic distortion is present, as evidenced by the "kink" at the bottom edges of the sine wave. This can be eliminated by filtering the output slightly, either with a post RC filter, or by adding a capacitor across the plate load resistor to act as a first-order lowpass filter to reduce a bit of the distortion. The filter cutoff frequency cannot be made too low at the plate resistor, or the gain will be reduced below the level necessary to sustain oscillation. A good first choice is to select a capacitor that will produce a -3dB point around 3 times the oscillation frequency. This can be calculated using the output impedance of the stage as follows:

\[ C = \frac{1}{2\pi f_0 R} = \frac{1}{2\pi \times 3 \times 7 \times 38.5K} = 0.197\mu F \]  

(use 0.2\mu F as the nearest standard value)

If the starting gain of the amplifier is too low, this extra filter may lower the gain too much to sustain oscillations, in which case it should be increased, or an RC post filter should be used after the oscillator.
The final schematic and output plot are shown below:

Completed Phase Shift Oscillator Design
Design modifications for a tremolo oscillator

There are several factors that must be taken into consideration when designing a phase shift oscillator for use as a tremolo oscillator in a guitar amplifier. They are:

- A tremolo circuit must have a variable frequency oscillator, which is adjustable by a single potentiometer. The oscillator needs to have a relatively wide frequency variation (typically from around 2Hz to 8Hz), the need to maintain a relatively constant amplitude over the entire adjustment range of the pot, and the need to keep the potentiometer value low enough to be able to use standard, off-the-shelf parts.
The ideal method of adjusting the frequency is to use a triple pot, to control all three phase shift sections. This is not always practical, so only one section is usually adjusted. It is usually best to adjust the last phase shift section, rather than the first one after the amplifier, as it will usually afford a wider range of control.

The requirement for lower value resistors forces the design to use larger capacitors, which have lower reactance at the frequency of oscillation. Because of this, it is better to design for a lower output impedance by using a tube with a lower internal plate resistance, or sacrifice some gain by using lower value plate resistances. This may, however, require a larger cathode bypass capacitor to maintain gain at lower frequencies.

When using the phase lead network, the impedance of the phase shift network is proportional to the value of R, so it should be made as large as practical, relative to the output impedance of the amplifier. This will minimize the gain variations as the frequency is adjusted, however, this is in contrast to the need to keep the potentiometer and resistance values low, so the best overall solution is to use the lowest practical output impedance.

- **The design procedure**

As in the above fixed-frequency design example, the first step is to design a suitable amplifier. For this example, we will assume a frequency adjustment range of 2Hz to 8Hz is desired. Note that the difference between the minimum frequency selected is very important at these low frequencies, because it determines the adjustment range of the oscillator. For example, a range from 1Hz to 7Hz requires an oscillator that can be adjusted over a 7:1 range, while a range of 2Hz to 8Hz only requires an oscillator that can be adjusted over a 4:1 range, even though the total frequency difference is the same. This makes a big difference in the design of the frequency adjustment portion of the oscillator.

- **Amplifier design:**
  - An amplifier with a gain of greater than 29 must first be designed. The requirement for wide range adjustability with low value potentiometers requires a low plate resistance tube. The 12AT7 is chosen for this example.

    Some typical 12AT7 numbers:

    plate resistance: ra = 10.9K
    amplification factor: mu = 60

  - A plate load resistance of 47K is chosen. Plotting this load line on the characteristic curves of a 12AT7 at 300V, results in a required grid bias point of around -3V/2.5mA, so a resistor of 3V/2.5mA = 1.2K is chosen. In order to maximize the gain, the cathode must be fully-bypassed at the frequency of oscillation. The midband gain for the 12AT7 with a fully-bypassed cathode is:

    \[ Av = \frac{mu 
    \times R_l}{R_l + ra} \]
    \[ = \frac{60 \times 47K}{47K + 10.9K} \]
    \[ = 48.7 \]

    Where:
    - \( R_l \) = the load resistor (47K in this example)
    - \( ra \) = the internal plate resistance (10.9K for a typical 12AT7)
    - \( mu \) = the mu of the tube (60 for a typical 12AT7)

    The calculated gain of 48.7 is higher than the required minimum gain of 29, so this amplifier will work in the phase shift oscillator.
Next, the output impedance of the stage is calculated. Since the internal plate resistance is effectively in parallel with the plate load resistor, the output impedance (for the signal taken off the plate) will be:

\[
R = \frac{1}{\frac{1}{\text{ra}} + \frac{1}{\text{Rl}}}
\]

\[
= \frac{1}{\frac{1}{10.9K} + \frac{1}{47K}}
\]

\[
= \frac{1}{8.85K}
\]

Next, the resistance seen looking into the cathode must be calculated, as it is needed to determine the value of bypass capacitor needed.

The resistance seen looking into the cathode (with \(R_k\) unbypassed) is:

\[
R_k' = \frac{(R_l+ra)}{(\mu+1)}
\]

\[
= \frac{(47K + 10.9K)}{(61)}
\]

\[
= 949 \text{ ohms}
\]

Therefore, the total cathode resistance is the parallel combination of the cathode resistance, \(R_k'\), and the cathode resistor, \(R_k\), as below:

\[
R = R_k' || R_k
\]

\[
= 949 || 1.2K
\]

\[
= 530 \text{ ohms}
\]

Next, the value of the cathode bypass capacitor must be calculated. Since the gain of the amplifier must be maximized at the lowest frequency of oscillation, the minimum capacitor value is chosen using the following equation for the -3dB point of the amplifier.

\[
f = \frac{1}{2\pi R C}
\]

Solving for \(C\):

\[
C = \frac{1}{2\pi R f}
\]

The minimum value of bypass capacitor is therefore:

\[
C = \frac{1}{2\pi 530 \times 2 \text{Hz}} = 150 \text{uF}
\]

The amplifier gain will be down -3dB at this point, corresponding to a gain decrease of 0.707 times 48.7, or 34, which is still above the required minimum of 29. However, in order to maximize gain and keep the phase shift associated with the cathode bypass capacitor to a minimum, the capacitor value should be increased to around five to ten times the minimum calculated value, such as 820uF, giving a -3dB point of 0.62Hz. Note that the required capacitor value can get rather large if low frequency oscillators are designed with tubes that have low gains and low internal plate resistances.
Phase shift network component selection:

- Since the impedance is proportional to the shunt element in the phase shift network\(^2\), in this case, the resistor, a suitable impedance value must first be chosen. The input impedance of the network must be large in comparison to the output impedance of the amplifier, so as not to load the output appreciably, which would reduce the gain, possibly to a point where it can no longer sustain oscillations. A good minimum value is around ten times the actual output impedance of the amplification stage. Since the input impedance is proportional to the shunt element, and is approximately twice the value of the shunt element at the oscillation frequency, the resistance can be chosen to be around half the required impedance. This resistance will then determine the value of capacitor necessary to achieve the desired frequency of oscillation. Since we have an output impedance of 8.85K in our example, a good minimum value for the input impedance is ten times this value, to prevent loading of the output stage. Since the resistance value to achieve this impedance is around half the total impedance, a value of five times the output impedance, or 5*8.85K = 44.3K, will work.

- Next, the capacitance value is calculated using the formula for the frequency:

\[
 f_0 = \frac{1}{2\pi \sqrt{6}; R \cdot C}
\]

solving for C, and using the value of the highest frequency (8Hz) in the oscillator range:

\[
 C = \frac{1}{2\pi \sqrt{6}; f_0 \cdot R} = \frac{1}{2\pi \cdot 8 \cdot 44.3K} = .183\mu F \quad (use \ 0.18\mu F \ as \ the \ nearest \ smaller \ standard \ value)
\]

Since capacitor values are more commonly available in 10% values, and resistors are more commonly available in 5% or even 1% values, the resistor value should be recalculated based on the standard capacitor value chosen.

solving for R:

\[
 R = \frac{1}{2\pi \sqrt{6}; f_0 \cdot C} = \frac{1}{2\pi \cdot 8 \cdot .18\mu F} = 45K \quad (use \ 47K \ as \ the \ nearest \ larger \ standard \ value)
\]

When choosing the capacitor, it is best to choose the next smaller size because this will make the input impedance larger, because it will require a larger resistance to achieve the desired frequency. Likewise, when choosing the resistance, it is best to choose the next larger size, as this will also increase the input impedance of the phase shift network.

- Next, in order to vary the frequency, the resistance must be varied over a range equivalent to the frequency range required. In this example, the range must be 4:1, so a triple pot of 4*47K, or 188K is required, so a 200K is chosen as the the nearest standard value. The triple pot is chosen in this example because it provides the best range of adjustment.

Following is the schematic of the finished circuit:
Completed Variable Phase Shift Oscillator Design

This design, using standard values, ended up with a frequency adjust range from 1.5 Hz to 7.3 Hz. It can be trimmed by decreasing the capacitor value from 0.18 uF to 0.15 uF to achieve the original specification of 2 Hz to 8 Hz. If desired, a lowpass filter can be made using a capacitor across the 47 K plate resistor as demonstrated in the fixed frequency design, but it will only be effective at the higher frequencies, as it would require a variable cutoff frequency filter to properly filter over the entire adjustment range of the oscillator.

Design considerations for using a single pot to control frequency

It is desirable from a parts cost and availability standpoint to be able to use a single pot to control frequency instead of a triple pot. While this will
cause a moderate amount of amplitude variation with frequency, this is not too much of a problem with a guitar amplifier because it is natural to adjust the speed and intensity interactively until the desired effect is achieved, as long as the variation isn't too noticeable. The main design considerations are:

- The amplifier needs to have higher gain than the nominal gain of 29. This is to account for gain variations as the impedance variation and gain of the network is greater with the single pot adjustment.
- The phase shift network should be designed for a frequency around half the maximum, and the pot should be used to set gains above and below the nominal frequency by using a smaller series frequency limit resistor than the calculated nominal value for the half frequency.
- The pot should have a range of no more than five times the nominal resistance value, otherwise the oscillations may stop at higher settings. The amount of frequency variation is limited to around 3:1 or so.

- The design procedure

As in the above fixed-frequency design example, the first step is to design a suitable amplifier. For this example, we will assume a frequency adjustment range of 2Hz to 6Hz is desired. This is in keeping with the approximate 3:1 range limit imposed by the single-pot design.

- **Amplifier design:**
  Since we need an amplifier with excess gain, we will use the 12AX7 amplifier design from the first fixed-frequency example. However, since we want a nominal frequency of around 4Hz, the phase shift components will be recalculated as follows:

- **Phase shift network component selection:**
  - The impedance of the phase shift network will remain the same as previously calculated, so the originally calculated resistor value of 200K will be used.
  - Since the frequency will be varied over a range lower than the original design spec of 7Hz, the cathode bypass capacitor value should be increased from 470uF.

  The minimum value of bypass capacitor is therefore:

  \[
  C = \frac{1}{2\pi \times f_0 \times R} = 147\text{uF}
  \]

  As indicated in the original design, this value of capacitor will result in the amplifier response being down -3dB at the lowest oscillation frequency of 2Hz. It is best to increase the capacitor value by a factor of five to ten to avoid the gain loss at the lower frequency range. A good compromise value between size and frequency response is 820uF, giving a -3dB point of 0.4 Hz.

  - Next, the capacitance value is calculated using the formula for the frequency:

    \[
    f_0 = \frac{1}{(2\pi \times \sqrt{6} \times R \times C)}
    \]

    solving for \( C \), and using the value of the nominal frequency (4Hz) in the oscillator range:

    \[
    C = \frac{1}{(2\pi \times \sqrt{6} \times 4 \times 200K)} = \frac{1}{(2\pi \times 2.45 \times 4 \times 200K)} = 0.081\text{uF} \text{ (use 0.082uF as the nearest standard value)}
    \]
Next, in order to vary the frequency, the resistance must be varied over a broader range than needed with the triple potentiometer version. In order to achieve a 3:1 ratio, the potentiometer range must be around 5:1, so a pot of 5*200K, or 1Meg, is required.

The potentiometer is connected in series between the first resistor and ground. In order to adjust the frequency both above and below the nominal value of 4Hz, the first resistor value is lowered from 200K to around 1/5 the value, or 40K (39K is chosen as the nearest standard value).

Following is the schematic of the finished circuit:
This oscillator has a range of 2Hz to 6.5Hz, with an amplitude variation from 204V at 6.5Hz to 163V at 2Hz. As expected, the amplitude variation is quite large, compared to the triple pot version, and the frequency adjustment range is smaller. However, this should be acceptable performance for a guitar amplifier tremolo oscillator.

**Design considerations: footswitch and startup issues for tremolo circuits**

The only way to get an oscillator like this to start rapidly and reliably is to introduce an outside interference, in the way of a voltage or current transient. In general, the larger the transient, the quicker the start. The typical method of turning the tremolo on and off is to use a footswitch that kills the biasing of the amplifier section of the oscillator.

One of the problems with this kind of footswitch circuit is that the amplifier is cut off by removing the bias on the cathode with the shorting switch. In order to oscillate, it has to first amplify. The footswitch provides the shock start in the form of a transient as the cathode voltage rises from the short to the 1.6V nominal stable point, but the bypass cap, which must be there in order to attain enough gain to oscillate, now works against things by slowing down and limiting the magnitude of the startup transient. As can be seen, the previously mentioned requirement for a large value bypass capacitor in order to get enough gain at low frequencies is now at odds with the requirement for a fast transient to start the oscillator reliably. In these cases, the capacitor needs to be designed for a value just above that required for reliable oscillation. In some cases, the oscillator will still be slow to start, or may not start at all.

It is usually easier to get these type oscillators to start if the amplifier stays on, biased to the proper point of operation, and an outside AC transient is introduced somewhere in the circuit. However, this is not so easy to do when all you have is a footswitch that must be grounded on one side, and have a safe voltage on the other terminal, and no circuit to generate a startup pulse to inject into the circuit.

One way to accomplish this is to add a large value resistor from the "center resistor", or R2 as shown in the above examples, to the power supply, and connect the footswitch to the junction of the two resistors. Typically the resistor should be around ten times the value of the center resistor. When the switch is grounded, the oscillator is off, because the AC feedback path is broken, but the DC bias on the tube remains the same, because the coupling caps on either side block the DC voltage. This leaves the amplifier biased properly for normal operation. When the switch is opened, there is a fast, relatively high voltage AC transient that is coupled into the grid circuit, which starts the oscillations rapidly. The drawback is that the junction of the two resistors is at about 1/11 the supply voltage normal operation, which means this voltage appears on the center terminal of the footswitch, which may not be safe, particularly if the center to ground resistor fails. Also, this circuit may not speed up the initial power-on startup delay, only the footswitch startup delay, but this is usually acceptable.

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**Appendix A: The math behind the phase shift oscillator:**

1. **Phase lead network analysis:**

   A single-section, phase lead network transfer function can be derived using the voltage divider rule as follows:
\[
V_o = V_i \frac{sR}{sRC + 1}
\]

Therefore, the transfer function, \( H(s) \), is equal to:

\[
H(s) = \frac{V_o}{V_i} = \frac{sRC}{sRC + 1}
\]

where \( s = jw = j2\pi f \)

and \( j = \sqrt{-1} \)

A complex number of the form \( C = A + jB \) has both a magnitude and a phase. The magnitude is equal to the square root of the sum of the squares of the real and imaginary parts, and the phase is equal to the arctangent of the imaginary part divided by the real part, as shown below:

\[
\text{Magnitude } H(s) = \sqrt{A^2 + B^2}
\]

\[
\text{Phase } H(s) = \tan^{-1}(B/A)
\]

Therefore, the magnitude of the phase lead transfer function is as follows:

\[
|H(jw)| = \sqrt{(wRC)^2/(1 + w^2RC^2)}
\]

And the phase of the phase lead transfer function is:

\[
\phi = (\text{phase of numerator} - \text{phase of denominator}) = \tan^{-1}(wRC/0) - \tan^{-1}(wRC/1) = 90^\circ - \tan^{-1}(wRC)
\]

Using the example given, \( R = 1\text{M} \) and \( C = 0.01\text{uF} \):

\[
|H(jw)| = \frac{(2\pi f)(0.01\text{e}^{-6})}{\sqrt{1 + (2\pi f)^2(0.01\text{e}^{-6})^2}} = \frac{0.06283\pi f}{\sqrt{1 + 0.00395f^2}}
\]

\[
\phi = 90 - \tan^{-1}(2\pi fR/C) = 90 - \tan^{-1}(0.06283f)
\]

Therefore, at a frequency of 15.9Hz, the magnitude and phase would be:

\[
|H(jw)| = \frac{0.06283\times15.9}{\sqrt{1 + 0.00395\times15.9^2}} = 0.707 = -3.01\text{dB}
\]

\[
\phi = 90 - \tan^{-1}(0.06283\times15.9) = 45^\circ
\]

If \( f = 0 \) is substituted into the equations, the resultant magnitude goes to zero (as it should, since the capacitor blocks DC) and the resultant phase goes to 90 degrees.

If \( f = \infty \) is substituted into the equations, the resultant magnitude goes to 1, or 0dB, and the resultant phase shift goes to zero.
A plot of the magnitude and phase of the transfer function can be made by substituting in values of $f$ and solving for the resulting magnitude and phase numbers.

It is important to note that the frequency at which a 60 degree phase shift occurs in a single phase lead section is not the same frequency at which a 180 degree phase shift occurs in a three section phase lead network, so you can't just solve the single-section phase equation for frequency, and plug in a value of 60 degrees to find the resultant frequency at which oscillation will occur for given values of $R$ and $C$. The procedure for determining frequency of oscillation for a three-section phase lead network is described in the next section on phase shift network analysis.

Phase shift network analysis:

The transfer function of the phase shift network can be determined as follows:

Using mesh analysis, and substituting general impedance variables $Z_1$ for $C$ and $Z_2$ for $R$, the following three equations can be derived:

1. $V_i = Z_1 I_1 + Z_2 (I_1 - I_2)$
2. $0 = Z_2 (I_2 - I_1) + Z_1 I_2 + Z_2 (I_2 - I_3)$
3. $0 = Z_2 (I_3 - I_2) + Z_1 I_3 + Z_2 I_3$

This can be rearranged and written in terms of the mesh currents as follows:

1. $V_i = (Z_1 + Z_2) I_1 - Z_2 I_2$
2. $0 = -Z_2 I_1 + (Z_1 + 2Z_2) I_2 - Z_2 I_3$
3. $0 = -Z_2 I_2 + (Z_1 + 2Z_2) I_3$

which gives the following matrix equation:

$$
\begin{bmatrix}
V_i \\
I_1 \\
I_2 \\
I_3
\end{bmatrix} =
\begin{bmatrix}
(Z_1 + Z_2) & -Z_2 & 0 \\
-Z_1 & (Z_1 + 2Z_2) & -Z_2 \\
0 & -Z_2 & (Z_1 + 2Z_2)
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
$$

This matrix can be solved for the individual currents by using Cramer's Method as follows:

First, the characteristic determinant of the matrix is calculated as follows:

$$
\text{Det} = \begin{vmatrix}
(Z_1 + Z_2) & -Z_2 & 0 \\
-Z_1 & (Z_1 + 2Z_2) & -Z_2 \\
0 & -Z_2 & (Z_1 + 2Z_2)
\end{vmatrix} = Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3
$$
Next, the individual currents can be solved by substituting the voltage matrix into the appropriate position in the numerator matrix and solving the resulting determinant, and dividing by the characteristic determinant as follows:

\[
\begin{vmatrix}
(Z_1 + Z_2) & -Z_2 & V_i \\
-Z_2 & (Z_1 + 2Z_2) & 0 \\
0 & -Z_2 & 0
\end{vmatrix} = V_i^* \frac{Z_2^2}{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}
\]

\[I_3 = \frac{V_i^*}{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}
\]

\[I_3\] was chosen first so the output voltage could be determined in order to derive the transfer function. Since \[I_3\] is now known, the output voltage is simply \[I_3\] multiplied by the last shunt impedance, \[Z_2\]:

\[V_o = V_i^* \frac{Z_2^2}{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}
\]

\[= V_i^* \frac{Z_2^3}{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}
\]

Therefore, the transfer function is:

\[\frac{V_0}{V_i} = \frac{Z_2^3}{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}
\]

This can be rewritten by dividing the numerator and denominator by \[Z_2^3\] as follows:

\[\frac{V_0}{V_i} = \frac{1}{Z_1^3Z_2^3 + 5Z_1^2Z_2^2 + 6Z_1Z_2 + 1}
\]
Now, substituting the variable 'x' for $Z_1/Z_2$,

$$V_o/V_i = \frac{1}{x^3 + 5x^2 + 6x + 1}$$

The criteria for oscillation is positive feedback, i.e., the total phase shift around the loop have to be equivalently zero (a multiple of 360 degrees) and a gain of unity or greater. This criterion can be met by a total phase shift through the network of 180 degrees, since the amplifier contributes 180 degrees to the phase shift through its inversion. In order for the phase shift to be 180 degrees, the imaginary parts of the transfer function must be zero. The squared term, $5x^2$ and the constant 1, are purely real, since the square of $j$ is equal to -1, so the following equation must be met for a zero imaginary part:

$$x^3 + 6x = 0$$

which gives:

$$x^2 = -6$$

so:

$$x = \pm \sqrt{-6} = \pm j\sqrt{6}$$

If $x^3 + 6x = 0$, the magnitude equation at the frequency of oscillation simplifies to:

$$V_o/V_i = \frac{1}{5x^2 + 1}$$

Substituting $x = -j\sqrt{6}$ into this equation for $x$ (-j because $Z_1$ is a capacitor) gives:

$$V_o/V_i = \frac{1}{5(-j\sqrt{6})^2 + 1} = \frac{1}{5(-1\times6) + 1} = \frac{-1}{29}$$
This means that the transfer function of the phase shift network has a gain of 1/29, and the negative sign indicates a phase inversion of 180 degrees. Therefore, in order to satisfy the gain criterion for oscillation, the amplifier must have a gain of -29 (once again, the negative sign indicating phase inversion of 180 degrees). The gain can be higher than this, but the distortion will be lowest at the minimum gain necessary to sustain oscillations.

The frequency of oscillation is also determined by the above oscillation criteria, as shown below:

Since \( x = \frac{Z_1}{Z_2} = \frac{(1/(jwC))/R}{1/(jwRC)} = -j^\sqrt{6} \),

\[
1 = \frac{1/(\sqrt{6}R*C)}{}
\]

Since \( w = \frac{2*Pi*f}{1/(\sqrt{6}R*C)} \),

\[
f = \frac{1/(2*Pi*sqrt(6)*R*C)}{}
\]

This is the frequency which will give a total phase shift through the network of -180 degrees, which will result in oscillation.

The input impedance of the network can be determined by first calculating \( I_1 \) with a \( V_i \) of 1V, then taking the reciprocal of it. This is done as follows:

\[
| 1 & -Z_2 & 0 | \quad \begin{array}{c} Z_1^2 + 4Z_1Z_2 + 3Z_2^2 \\ Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3 \end{array}
| 0 & (Z_1 + 2Z_2) & -Z_2 | \quad \begin{array}{c} Z_1^2 + 4Z_1Z_2 + 3Z_2^2 \\ Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3 \end{array}
| 0 & -Z_2 & (Z_1 + 2Z_2) | \quad \begin{array}{c} Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3 \end{array}
\]

Taking the reciprocal to get the input impedance gives:

\[
Z_{in} = \frac{Z_1^3 + 5Z_1^2Z_2 + 6Z_1Z_2^2 + Z_2^3}{Z_1^2 + 4Z_1Z_2 + 3Z_2^2}
\]

Dividing both numerator and denominator by \( Z_2^3 \), after multiplying the denominator and the entire equation by \( Z_2/Z_2 \), (to make the function proper in terms of degree of exponents in the numerator and denominator) gives:

\[
Z_{in} = \frac{Z_1^3/Z_2^3 + 5Z_1^2Z_2/Z_2^3 + 6Z_1Z_2^2/Z_2^3 + Z_2^3/Z_2^3}{Z_1^2Z_2/Z_2^3 + 4Z_1Z_2^2/Z_2^3 + 3Z_2^3/Z_2^3}
\]

\[ Z_2 \]
This simplifies to:

\[
Z_{\text{in}} = Z_2 \times \frac{Z_1^3/Z_2^3 + 5Z_1/Z_2^2 + 6Z_1/Z_2 + 1}{Z_1^2/Z_2^2 + 4Z_1/Z_2 + 3}
\]

Substituting the variable 'x' for \(Z_1/Z_2\),

\[
Z_{\text{in}} = Z_2 \times \frac{x^3 + 5x^2 + 6x + 1}{x^2 + 4x + 3}
\]

At the frequency of oscillation, \(Z_{\text{in}}\) becomes:

\[
Z_{\text{in}} = Z_2 \times \frac{(-j\sqrt{6})^3 + 5(-j\sqrt{6})^2 + 6(-j\sqrt{6}) + 1}{(-j\sqrt{6})^2 + 4(-j\sqrt{6}) + 3}
\]

Simplified, this becomes:

\[
Z_{\text{in}} = \frac{-29Z_2}{-3 - 4j\sqrt{6}} = \frac{-29Z_2}{-3 - j9.798} = \frac{Z_2}{0.103 + j0.338}
\]

This impedance equation indicates that the input impedance at the oscillation frequency is proportional to \(Z_2\), but not \(Z_1\). If the frequency is to be varied, this impedance must remain constant, or the amplitude of the oscillations will vary. If the gain of the amplifier is set to the critical value of 29, and the impedance decreases, the gain will drop and the oscillations will decay to zero. Likewise, if the impedance increases, the gain will increase, and the oscillation amplitude will increase, and there will also be an increase in the distortion in the output. Therefore, the best method of varying the frequency of oscillation is to vary all three \(Z_1\) impedances simultaneously. This will require a triple-gang variable capacitor, which is not practical at the low frequencies involved in audio oscillators used in tremolo circuits. A better approach would be to use the phase lag network version, and vary the three \(Z_1\) impedances simultaneously with a triple-gang potentiometer, since they are resistances in the phase lag configuration. This requires an extra isolation capacitor as well as an extra bias resistor, which must be large in relation to the series resistance, to avoid loading the network. In addition, the phase lag version has much higher distortion of the sine wave at the plate output. Alternately, the gain of the amplifier could be made much larger than 29, and the output will have amplitude variations as well as distortion of the sine wave as the amplitude is adjusted. The amplitude variation is not too much of an issue with a guitar amplifier, because it is normal to set the speed and then adjust the intensity to the desired level. Also, the amplitude variations can be minimized by making the phase shift network impedance much larger than the output impedance of the amplifier stage.